

# GUTS AND VOLUME FOR HYPERBOLIC 3-ORBIFOLDS WITH UNDERLYING SPACE $S^3$

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**ABSTRACT.** For a hyperbolic 3-orbifold with underlying space the 3-sphere, we obtain a lower bound on its volume in the case that it contains an essential 2-suborbifold with underlying space the 2-sphere with four cone points. Our techniques involve computing the guts of the orbifold split along the 2-suborbifold via a careful analysis of its topology. We also characterize the orbifolds of this type that have empty guts.

## 1. INTRODUCTION

This paper contributes to understanding the organization of the volumes of hyperbolic 3-manifolds and 3-orbifolds. One common theme in this organization is the classification of such spaces for which the presence of a particular type of 2-dimensional sub-object informs the volume of the ambient space. In the case we are considering, an embedded incompressible 2-suborbifold in a 3-orbifold (one measure of the 3-orbifold's topological complexity) either informs us about the volume of this 3-orbifold or about its topological structure. Examples of this volume/complexity dynamic occur in recent years in the work of Gabai, Meyerhoff and Milley in the identification of the Weeks–Fomenko–Matveev manifold as the lowest volume hyperbolic 3-manifold [9, 10]. Similarly, Gehring, Marshall, and Martin have identified the lowest volume hyperbolic 3-orbifolds [11, 15] based, in part, on this idea. Miyamoto gives a lower volume bound for hyperbolic 3-manifolds [16] (generalized to 3-orbifolds in [20]) with totally geodesic boundary. Recent work by a subset of the authors identifies the lowest volume polyhedral hyperbolic 3-orbifolds that contain an arbitrary essential 2-suborbifold [3].

In the current paper, we employ a result of Agol, Storm, and Thurston [2] to find lower volume bounds (or else, a topological characterization)

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for a large class of hyperbolic 3-orbifolds that contain a particular type of essential 2-suborbifold (i.e. *Haken* 3-orbifolds) in terms of the topology of that suborbifold.

Let  $\text{Vol}(\cdot)$  denote hyperbolic volume,  $\chi(\cdot)$  Euler characteristic, and  $V_8 \approx 3.66$  the volume of the regular, ideal, hyperbolic octahedron. Let  $S^2(n_1, n_2, n_3, n_4)$  denote the orientable 2-orbifold with base space  $S^2$  and cone points of orders  $n_1, n_2, n_3$ , and  $n_4$ . We prove the following theorem:

**Theorem 1.1.** *Let  $\mathcal{O}$  be a compact, orientable, irreducible, turnover-reduced, 3-orbifold with underlying space  $S^3$  and singular set  $\Sigma$  whose interior admits a hyperbolic structure of finite volume. Suppose  $\mathcal{O}$  contains an incompressible 2-suborbifold  $\mathcal{S}$  of type  $S^2(n_1, n_2, n_3, n_4)$ . Then one of the following lower bounds for  $\text{Vol}(\mathcal{O})$  holds:*

- (1)  $\text{Vol}(\mathcal{O}) \geq -V_8\chi(\mathcal{S}) = V_8(2 - \sum_{i=1}^4 1/n_i)$ , or
- (2)  $\text{Vol}(\mathcal{O}) \geq \frac{1}{2}V_8(-\chi(\mathcal{S}) + 1 - 1/n_{i_1} - 1/n_{i_2})$ ,  
(where  $\{n_{i_1}, n_{i_2}\} \subset \{n_1, n_2, n_3, n_4\}$ ), or
- (3)  $\text{Vol}(\mathcal{O}) \geq -\frac{1}{2}V_8\chi(\mathcal{S})$ , or
- (4)  $\text{Vol}(\mathcal{O}) \geq \frac{1}{2}V_8(2 - 1/n_{i_1} - 1/n_{i_2} - 1/n_{i_3} - 1/n_{i_4})$ ,  
(where  $\{n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}\} \subset \{n_1, n_2, n_3, n_4\}$ ), or
- (5)  $\text{Vol}(\mathcal{O}) \geq \frac{1}{2}V_8(1 - 1/n_{i_1} - 1/n_{i_2})$ ,  
(where  $\{n_{i_1}, n_{i_2}\} \subset \{n_1, n_2, n_3, n_4\}$ ), or
- (6)  $\mathcal{O}$  has one of the forms given in Section 4.1.

Note that we mean something more general than the standard use of hyperbolic structure of finite volume. We allow not only for cusps with Euclidean cross-sections, but for totally geodesic boundary components. See the end of Section 2.1 for details.

Let  $D_*^2(n_1, n_2)$  denote the nonorientable 2-orbifold with base space  $D^2$  with mirrored boundary and with interior containing cone points of orders  $n_1$  and  $n_2$ . Using the fact that an incompressible 2-suborbifold in  $\mathcal{O}$  of this form has a regular neighborhood whose boundary is an orientable and incompressible  $S^2(n_1, n_1, n_2, n_2)$ , the next result follows immediately from the above theorem:

**Corollary 1.2.** *Under the same conditions on  $\mathcal{O}$  as above, suppose  $\mathcal{O}$  contains an incompressible 2-suborbifold  $\mathcal{S}$  of type  $D_*^2(n_1, n_2)$ . Then one of the following lower bounds for  $\text{Vol}(\mathcal{O})$  holds:*

- (1)  $\text{Vol}(\mathcal{O}) \geq -V_8\chi(\mathcal{S}) = V_8(1 - 1/n_1 - 1/n_2)$ , or
- (2)  $\text{Vol}(\mathcal{O}) \geq \frac{1}{2}V_8(1 - 1/n_{i_1} - 1/n_{i_2})$ ,  
(where  $\{n_{i_1}, n_{i_2}\} \subset \{n_1, n_2\}$ ), or
- (3)  $\mathcal{O}$  has one of the forms given in Section 4.1.

Note that the smallest possible lower bounds given by Theorem 1.1 and Corollary 1.2 is  $\frac{1}{12}v_8 \approx 0.305$ . This volume bound occurs in number (5) of Theorem 1.1 and in number (2) of Corollary 1.2 when  $n_{i_1} = 2$  and  $n_{i_2} = 3$ . As a consequence, one can conclude that under the hypotheses, if  $\text{Vol}\mathcal{O} < \frac{1}{12}v_8$ , then  $\mathcal{O}$  has empty guts and is of the form described in Proposition 4.2.

**1.1. Organization.** In Section 2, we give the relevant background on orbifolds and describe the conventions that we will use throughout the paper. We also describe the characteristic suborbifold theory and relevant ideas. In Section 3 we prove a lemma that describes how essential annuli can arise in certain 3-orbifolds and prove the main theorem of the paper. In Section 4, we characterize orbifolds having empty guts.

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## 2. BACKGROUND AND DEFINITIONS

We recall some necessary facts about orbifolds here, and refer the reader to several excellent resources [4, 8]. An  $n$ -orbifold is a generalization of the notion of  $n$ -manifold that allows for local neighborhoods to be modeled on the quotient of  $\mathbb{R}^n$  by a (possibly trivial) finite group acting properly discontinuously. In many cases,  $n$ -orbifolds arise as the quotient of a manifold by a finite group of symmetries.

**2.1. Orbifolds.** An orientable 3-orbifold  $\mathcal{O}$  is a pair  $(X_{\mathcal{O}}, \Sigma_{\mathcal{O}})$  where  $X_{\mathcal{O}}$  is an orientable 3-manifold and  $\Sigma_{\mathcal{O}}$  is an embedded graph in  $X_{\mathcal{O}}$  with edges labeled by integers  $n \geq 2$ . The manifold  $X_{\mathcal{O}}$  is called the *underlying space* of the orbifold  $\mathcal{O}$ . The graph  $\Sigma_{\mathcal{O}}$  is called the *singular locus* of the orbifold. The graph  $\Sigma_{\mathcal{O}}$  need not be connected and may contain components consisting of single loops (while multi-edges between vertices are allowed, they tend to violate the geometric assumptions we will use in this paper). In the case where  $X_{\mathcal{O}}$  is closed,  $\Sigma_{\mathcal{O}}$  must be a trivalent graph. If  $X_{\mathcal{O}}$  has boundary, then  $\Sigma_{\mathcal{O}}$  may also have univalent vertices on the boundary of  $X_{\mathcal{O}}$ . The labeling on an edge of the singular locus is called the *order* of the singular locus along that edge.

The data  $(X_{\mathcal{O}}, \Sigma_{\mathcal{O}})$  completely describe the orbifold. Neighborhoods  $U \subset (X_{\mathcal{O}} \setminus \Sigma_{\mathcal{O}})$  of points are modeled on  $\mathbb{R}^3$ . Neighborhoods  $U_x$  of points  $x \in \Sigma_{\mathcal{O}}$  are modeled on  $\mathbb{R}^3/G_x$  where  $G_x$  is a finite subgroup

of  $O(3)$ . In the case where  $x$  lies in an edge of  $\Sigma$  labeled  $n$ ,  $G_x \approx \mathbb{Z}_n$ . If  $x$  is a vertex of  $\Sigma$  meeting edges labeled  $p$ ,  $q$ , and  $r$ , then  $G_x$  is the spherical triangle group generated by rotations of order  $p$ ,  $q$ , and  $r$ . Note that this implies that  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ .

We will abuse notation in this paper and allow for the labeling of the singular locus to violate the  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$  condition at a vertex. We will refer to a vertex of the singular locus as a *spherical, rigid Euclidean, or hyperbolic vertex* if  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$  is greater than, equal to, or less than 1, respectively. A hyperbolic or rigid Euclidean vertex encodes a boundary component of the orbifold. If  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ , then the orbifold has a boundary component obtained by doubling a triangle with angles  $\pi/p$ ,  $\pi/q$ , and  $\pi/r$  with edges of the singular locus meeting the corresponding cone points on this boundary component. The reason for this abuse of notation is that it turns out that these “vertices” can usually be treated the same, regardless of whether they are spherical, rigid Euclidean, or hyperbolic. For this reason, when we say that an orbifold has underlying space  $S^3$ , it may be the case that there are boundary components coming from the non-spherical vertices.

**2.2. Suborbifolds.** We say that  $\mathcal{S}$  is an *orientable 2-dimensional sub-orbifold* of an orientable 3-orbifold  $\mathcal{O}$  if  $\mathcal{S}$  is an embedded, orientable topological surface such  $\mathcal{S} \cap \Sigma$  is either empty or is a 0-dimensional subset of the edges of  $\Sigma$ . Such an  $\mathcal{S}$  may be given the structure of a 2-orbifold. The *underlying space* of the orbifold structure is the topological surface itself. The *singular locus* is the collection of points of  $\mathcal{S} \cap \Sigma$  labeled by the same integers as  $\Sigma$ . These points are locally modeled on  $\mathbb{R}^2$  modulo the action of rotation of the corresponding order. A *nonorientable 2-suborbifold*  $\mathcal{S}$  of  $\mathcal{O}$  is an embedded topological surface in  $\mathcal{O}$  such that the boundary of a regular neighborhood of  $\mathcal{S}$  is a connected, orientable 2-suborbifold of  $\mathcal{O}$ . The singular locus of a nonorientable 2-suborbifold includes points labeled by integers along with arcs along which  $\mathcal{S}$  is locally modeled on  $\mathbb{R}^2$  modulo the action of reflection.

Just as in the case of manifolds, orbifolds have fundamental groups denoted by  $\pi_1(\cdot)$  (corresponding to the groups acting on their universal covers), and a 2-suborbifold of a 3-orbifold is called *incompressible* if its fundamental group injects into the fundamental group of its ambient space and *essential* if it is incompressible but not isotopic to a boundary component.

A *turnover* is a 2-suborbifold with underlying space a 2-sphere that meets  $\Sigma$  in three points. These turnovers are called *spherical, Euclidean*

(*rigid*), or *hyperbolic* if their three point labels satisfy (respectively) the above conditions for the correspondingly-named vertices of  $\Sigma$ . A *turnover-reduced* 3-orbifold is one in which every embedded turnover encloses the cone on a vertex of  $\Sigma$ , and a 3-orbifold is *irreducible* when every embedded 2-suborbifold that is the quotient of a 2-sphere by a discrete group bounds a quotient of a 3-ball by the same discrete group.

One key definition we will use involves splitting a 3-orbifold along an embedded 2-suborbifold. This yields an orbifold with boundary, in the sense mentioned in Subsection 2.1. We denote by  $\mathcal{O} \setminus \mathcal{S}$  the path metric completion of  $\mathcal{O} \setminus \mathcal{S}$ . If  $\mathcal{S}$  is an embedded orientable 2-suborbifold of  $\mathcal{O}$ , then  $\mathcal{O} \setminus \mathcal{S}$  consists of two 3-orbifolds with boundary equal to  $\mathcal{S}$ .

When a 3-orbifold has rigid and hyperbolic vertices, we consider these as boundary components by deleting an open neighborhood of the vertices. In this case, we say a 3-orbifold has a *hyperbolic structure* if deleting the rigid boundaries yields a space with a complete hyperbolic geometric structure in which each rigid vertex becomes a Euclidean (rigid) cusp and in which the hyperbolic vertices become totally geodesic boundary components in the geometric structure.

Important 2-suborbifolds we will consider are orbifold tori and annuli, which are best thought of as quotients of standard tori and annuli by symmetries. Figure 1 depicts the different types of orbifold annuli. In the figure, the symbol  $*$  indicates a part of the orbifold that has the quotient structure associated to a reflection. These annuli can occur in a 3-orbifold in many ways. As an example, consider Figure 3(i), with the lone singular loop labeled by  $e$  set equal to 2: an orbifold annulus as in the lower left of Figure 1 appears as the annular region between this loop and the equator of the spherical boundary of the orbifold depicted, with the edge labeled 2 acting as an orbifold mirror for the annulus. An (orientable) orbifold torus is either a topological torus or a topological 2-sphere with four singular points each labeled 2.

If a 2-suborbifold is not incompressible, it is called *compressible*. In our case, this implies the existence of an orbifold disk (a quotient of a disk by a symmetry) whose boundary lies on the 2-orbifold but that does not also bound an orbifold disk on the 2-orbifold. Examples of compressibility in the case of annuli are shown in Figure 2 (with the compressing disks in grey).

An *orbifold rational tangle* is the space obtained by taking either (1) a solid ball with two unknotted, integer-labeled strands that lie in its interior except at their endpoints, which lie on the boundary of the ball; or (2) a solid ball with five unknotted, integer-labeled strands that lie in its interior in the shape of an “H,” except for the four endpoints of the

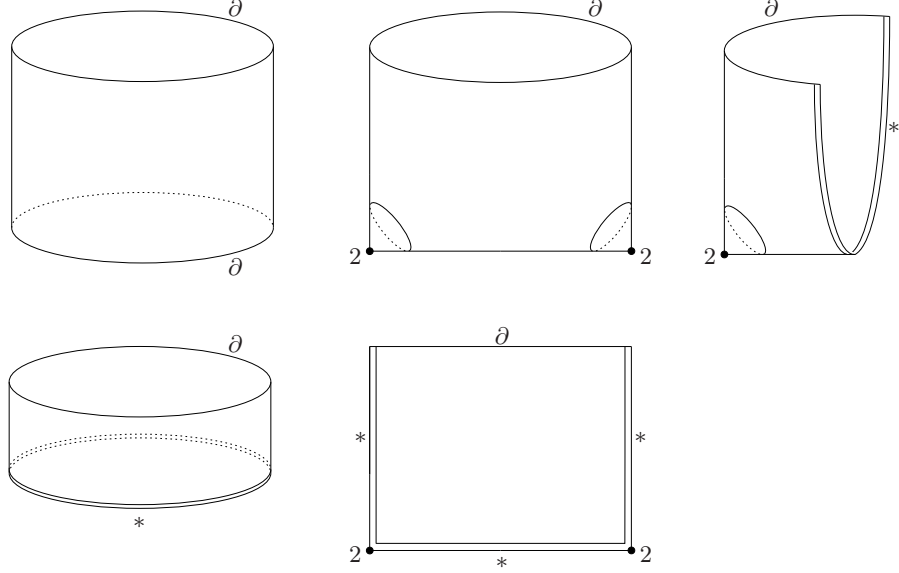


FIGURE 1. The different types of orbifold annuli. The symbols  $\partial$  and  $*$  indicate orbifold boundary and orbifold mirrors, respectively.

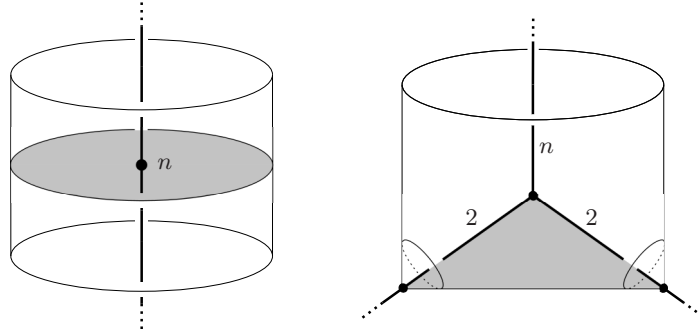


FIGURE 2. Orientable compressible Annuli. In both cases,  $n \geq 1$ .

“H,” which lie on the boundary of the ball; and performing an isotopy of the boundary of the ball that permutes the four singular points. If we assume a 3-orbifold  $\mathcal{O}$  satisfies the hypotheses of Theorem 1.1, then the turnover reducibility of  $\mathcal{O}$  implies the following simple proposition.

**Proposition 2.1.** *Let  $\mathcal{O}$  be a compact, orientable, irreducible, turnover-reduced, 3-orbifold with underlying space  $S^3$ . Suppose that  $\mathcal{S} = S^2(n_1, n_2, n_3, n_4)$  or  $\mathcal{S} = D_*^2(n_1, n_2)$  is a suborbifold of  $\mathcal{O}$ . Then  $\mathcal{S}$  is incompressible in  $\mathcal{O}$  if and only if no component of the  $\mathcal{O} \setminus \mathcal{S}$  is an orbifold rational tangle.*

*Proof.* Suppose  $\mathcal{S} = S^2(n_1, n_2, n_3, n_4)$  is compressible. Then a non-trivial loop  $C$  on  $\mathcal{S}$  bounds an orbifold disk  $D$  in (at least) one component of  $\mathcal{O} \setminus \mathcal{S}$ . Because  $C$  is non-trivial on  $\mathcal{S}$ ,  $C$  separates  $\mathcal{S}$  into two disks, each with two singular points of  $\mathcal{S}$ . Attaching  $D$  to these disks and using the irreducibility/turnover-reducibility of  $\mathcal{O}$  implies that this component of  $\mathcal{O} \setminus \mathcal{S}$  is an orbifold rational tangle. If one of the components of  $\mathcal{O} \setminus \mathcal{S}$  is an orbifold rational tangle, then this gives a compression of  $\mathcal{S}$  in  $\mathcal{O}$ .

If  $\mathcal{S} = D_*^2(n_1, n_2)$ , then the same proof works, using the boundary of a regular neighborhood of  $\mathcal{S}$ .  $\square$

**2.3. Pared orbifolds and guts.** We now introduce the definitions that will be central to the proof of our theorem.

**Definition 2.2.** A *pared orbifold* is a pair  $(\mathcal{O}, P)$ , where  $\mathcal{O}$  is a compact, orientable, irreducible 3-orbifold and  $P \subset \partial\mathcal{O}$  is a union of essential orbifold annuli and tori (possibly empty) such that

- (1) every abelian, noncyclic subgroup of  $\pi_1(\mathcal{O})$  is conjugate to a subgroup of the fundamental group of a component of  $P$ , and
- (2) every map of an orbifold annulus  $(F, \partial F) \rightarrow (\mathcal{O}, P)$  that is  $\pi_1$ -injective deforms, as a map of pairs, into  $P$ .

$P$  is called the **parabolic locus** of  $(\mathcal{O}, P)$ , and we define  $\partial_0\mathcal{O}$  to be  $\partial\mathcal{O} - \text{int}(P)$ .

Thurston proved ([4, Theorem 6.5], [5, 7, 14, 18, 19]) that an oriented, turnover-reduced pared orbifold with nonempty boundary (at least one component of which is not a hyperbolic turnover) admits a geometrically finite hyperbolic structure. The characteristic subpair theory for 3-manifolds [12, 13] holds in the category of 3-orbifolds [6]. In particular ([4, Remarks following Theorem 4.17], [17, Section 11, page 88]), we have the following characterization of the components of the characteristic suborbifold:

**Characteristic Suborbifold 2.3.** If  $(\mathcal{O}, P)$  is a pared orbifold with incompressible  $\partial_0\mathcal{O}$ , then there is a subpair  $(N, S) \subset (\mathcal{O}, \partial_0\mathcal{O})$ , which is a (possibly disconnected) suborbifold, uniquely determined up to isotopy of pairs and whose components are of the following three types, up to homeomorphism:

- (1)  $(T \times I, \emptyset)$ , a neighborhood of an orbifold torus component  $T \subset P$  (in our case, a neighborhood of a rigid cusp), or
- (2)  $(R, A)$ , where  $R$  is an orbifold solid torus and  $A$  is a nonempty union of essential orbifold annuli in  $\partial R$ , or



- (3) ( $I$ -bundles,  $\partial I$ -subbundles), with any  $I$ -bundles in the boundary of this type of component consisting of orbifold annuli that are not parallel into  $\partial\mathcal{O}$ .

For each component of type (2) above,  $\overline{\partial R - A}$  is a union of essential orbifold annuli in  $(\mathcal{O}, \partial_0\mathcal{O})$ . We thicken these and consider them as  $I$ -bundles over the annuli with their associated  $\partial I$ -subbundles lying in  $\partial_0\mathcal{O}$ , and add them to the components of type (3) to form an ( $I$ -bundle,  $\partial I$ -subbundle) pair  $(W, \partial_0W) \subset (\mathcal{O}, \partial_0\mathcal{O})$  called the *window* of  $(\mathcal{O}, \partial_0\mathcal{O})$ . The frontier of the window  $\partial_1W = \overline{\partial W - (\partial_0W \cup P)}$  consists of essential annuli in  $(\mathcal{O}, \partial_0\mathcal{O})$ . Following Agol [1, page 3275], we observe that the pair

$$(\overline{\mathcal{O} - W}, \overline{\partial(\mathcal{O} - W) - \partial_0\mathcal{O}})$$

is a pared suborbifold of  $\mathcal{O}$  whose parabolic locus consists of these essential annuli (up to isotopy). We have the following:

**Definition 2.4.** *The pared suborbifold  $(\overline{\mathcal{O} - W}, \overline{\partial(\mathcal{O} - W) - \partial_0\mathcal{O}})$  is denoted  $\mathbf{guts}(\mathcal{O}, P)$ . If  $\mathcal{O}$  is a compact, orientable 3-orbifold whose interior admits a hyperbolic metric of finite volume, and  $(X, \partial X) \subset (\mathcal{O}, \partial\mathcal{O})$  is an essential 2-orbifold, then we define  $\mathbf{guts}(X)$  to be  $\mathbf{guts}(\mathcal{O} \setminus X, \partial\mathcal{O} \setminus \partial X)$ .*

We note that if  $\partial_1W = \overline{\partial W - (\partial_0W \cup P)}$  denotes the frontier of the window of  $(\mathcal{O}, \partial_0\mathcal{O})$ , and if  $L$  is a component of  $\overline{\mathcal{O} - W}$ , then either

- $L$  is a solid orbifold torus and  $\partial_1W \cap L$  contains at least one essential orbifold annulus on  $\partial L$ , or
- all essential orbifold annuli in  $(L, \overline{\partial_0\mathcal{O} \cap L})$  are parallel into

$$(\partial_1W \cap L, \partial(\partial_1W \cap L)).$$

Components of the latter type are called *pared acylindrical orbifolds*, and they admit hyperbolic metrics with totally geodesic boundary.

A key tool we will use in this paper is the following (abbreviated here from the complete, more powerful) result of Agol, Storm, and Thurston, in combination with a result of Miyamoto, applied in the orbifold category (cf. [2, Theorem 9.1], [16]):

**Theorem 2.5.** *Let  $\mathcal{O}$  be a compact 3-orbifold with interior a hyperbolic orbifold of finite volume, and let  $\mathcal{S}$  be an embedded incompressible 2-suborbifold in  $\mathcal{O}$ . Then*

$$\text{Vol}(\mathcal{O}) \geq -V_8 \chi(\mathbf{guts}(\mathcal{S})) = -\frac{V_8}{2} \chi(\partial(\mathbf{guts}(\mathcal{S}))).$$



Accordingly, our strategy for the proof will be to identify the essential annuli in  $\mathcal{O} \setminus \mathcal{S}$  in order to effectively describe  $\text{guts}(\mathcal{S})$  and apply the volume bound of Theorem 2.5.

### 3. PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem. Before doing so, we need a technical lemma that describes how orientable, essential orbifold annuli can arise in certain 3-orbifolds. To prove the main theorem, this lemma will be used to describe the possibilities for  $\text{guts}(\mathcal{S})$ , where  $\mathcal{S}$  is an incompressible 2-suborbifold of an orbifold as described in the hypotheses of Theorem 1.1.

**Lemma 3.1.** *Let  $\mathcal{Q}$  be a compact, orientable, irreducible, atoroidal, turnover-reduced 3-orbifold with underlying space  $D^3$ . Assume  $\mathcal{Q}$  has four singular points on  $\partial D^3$  (labeled by their corresponding orders as  $a, b, c$ , and  $d$ ), and let  $\mathcal{S}$  denote the 2-orbifold  $\partial D^3$  together with these four points. Let  $P$  be the union of the rigid, Euclidean boundary components and let  $\partial_0 \mathcal{Q} = \partial \mathcal{Q} - P$ .*

*Then at most one of the following holds:*

- (1)  $(\mathcal{Q}, \partial_0 \mathcal{Q})$  contains a single, essential, nonsingular annulus,
- (2)  $(\mathcal{Q}, \partial_0 \mathcal{Q})$  contains a pair of essential, non-parallel  $D^2(2, 2)$  orbifold annuli, or
- (3)  $(\mathcal{Q}, \partial_0 \mathcal{Q})$  contains a single essential,  $D^2(2, 2)$  orbifold annulus.

*Moreover, the essential annuli are configured as in Figure 3.*

Note that this lemma doesn't preclude the presence of non-orientable essential orbifold annuli. However, the boundary of the regular neighborhood a non-orientable orbifold annulus is an orientable annulus and will be in one of the three cases described in the lemma.

*Proof.* There are 5 types of orbifold annuli to consider, as in Figure 1. The only components of  $\partial_0 \mathcal{Q}$  are  $\mathcal{S}$  and any hyperbolic turnovers corresponding to the boundary components of  $\mathcal{Q}$ . Because any closed loop on a hyperbolic turnover bounds an orbifold disk on the turnover, and because hyperbolic turnovers cannot have more than one cone point of order 2, the irreducibility of  $\mathcal{Q}$  implies that no essential annulus can have any part of its boundary carried in the hyperbolic turnover boundary components of  $\mathcal{Q}$ . So we need only consider essential annuli with boundary contained in  $\mathcal{S}$ .

Let  $A$  be such an essential annulus. If  $A$  is nonsingular, then its two boundary circles must consist of parallel curves on  $\mathcal{S}$  that separate  $\mathcal{S}$  into three regions: One annulus between the two components of  $\partial A$ , and, without loss of generality, two disks containing  $\{a, b\}$  and

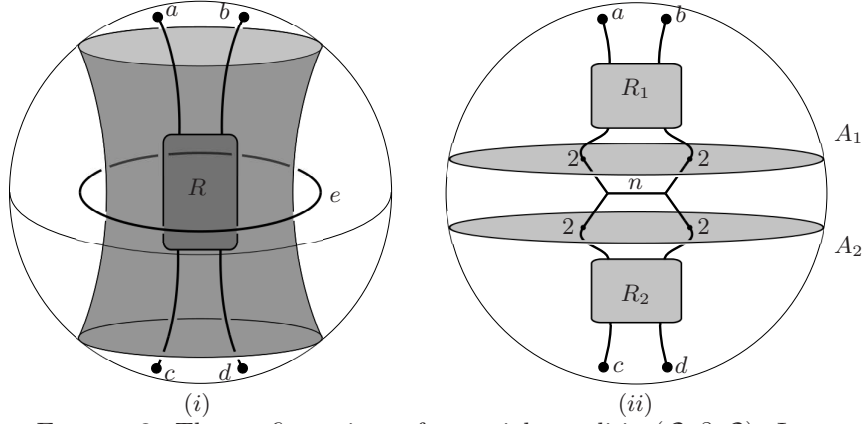


FIGURE 3. The configurations of essential annuli in  $(\mathcal{Q}, \partial_0 \mathcal{Q})$ . In (i), neither of the sets  $\{a, b\}$ ,  $\{c, d\}$  can equal  $\{2\}$ . In (ii), without loss of generality,  $\{c, d\}$  cannot equal  $\{2\}$ , and  $n \geq 2$  unless  $\{a, b\} = \{2\}$ , in which case  $n = 1$ . The essential annuli in (ii) are  $A_1$  and  $A_2$ . If  $n = 1$ , then  $A_1$  and  $A_2$  are parallel.

$\{c, d\}$ , respectively. (See Figure 3(i) as an aid to this discussion.) The two disks together with  $A$  bound a topological ball  $B$ , and the annulus together with  $A$  bounds a topological solid torus  $T$ . Because  $A$  is essential, some subset of  $\Sigma$  must lie in  $T$  (and similarly for the box marked  $R$  in the figure). However, since  $\mathcal{Q}$  is atoroidal, the only possibility is that one single loop of  $\Sigma$  lies at the core of  $T$  (labeled  $e$  in the figure). Notice that, once  $A$  has been identified, it is not possible for another nonsingular essential annulus (separating, for instance,  $\{a, d\}$  from  $\{b, c\}$  on  $\mathcal{S}$ ) to exist. See Figure 4. This is because such an annulus would force, by the same argument above, the existence of another loop in  $\Sigma$  (shown winding from north to south, in the figure) that would have to be contained in  $B$  but *also* contained in a solid torus (similar to  $T$  above, but with one longitudinal annulus in its boundary contained in  $\mathcal{S}$  and separating  $\{a, c\}$  from  $\{b, d\}$ ) that retracts to  $\mathcal{S}$  without its core curve having to cross through the core of  $T$  (labeled  $e$  in the figure). We note, further, that at least one integer from each of the pairs  $\{a, b\}$  and  $\{c, d\}$  must be greater than 2. If not, then it is possible to form an essential torus in  $\mathcal{Q}$ , contradicting the fact that  $\mathcal{Q}$  is atoroidal. See Figure 5. This argument also rules out the existence of any essential singular annulus (i.e., a disk with one or two order two singular points), and justifies Figure 3(i).

Notice that if the core curve  $e$  of  $T$  is labeled by 2, then we obtain another, nonorientable essential annulus, as described in Section 2 (in Figure 3(i), with its one boundary curve the “equator” of  $\mathcal{S}$  and its one mirrored edge along the curve labeled  $e$ ). We note, for later, that

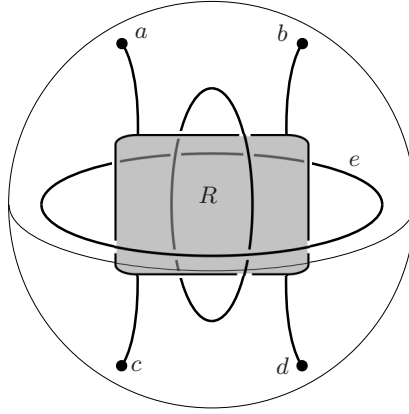


FIGURE 4. The two closed curves linking the region labeled  $R$  can be used to show that there can only be one essential nonsingular annulus.

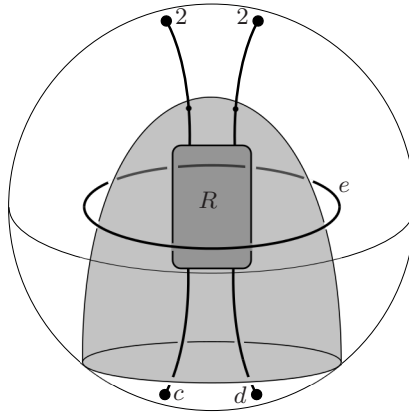


FIGURE 5. The indicated annular disk with two singular points determines an essential torus in  $\mathcal{O}$ .

the boundary of a regular neighborhood of this annulus is an essential orientable annulus, as described in the first part of this proof, that separates the regular neighborhood from the rest of  $\mathcal{Q}$  as an  $I$ -bundle.

If there is no nonsingular annulus, then there may exist up to two essential annuli, each of which with underlying space a disk and with two cone points of order 2. Because  $\mathcal{Q}$  is irreducible, the boundary of each such annulus must separate the cone points of  $\mathcal{S}$  into pairs, say, without loss of generality,  $\{a, b\}$  from  $\{c, d\}$ . (Note: for topological reasons, just as in the argument accompanying Figure 4, if there are two such annuli, then they must both separate  $\{a, b\}$  from  $\{c, d\}$ .) Because  $\mathcal{S}$  is not an orbifold torus (this would violate either its incompressibility or the fact that  $\mathcal{Q}$  is atoroidal), then supposing such annuli exist leads

to two possibilities: neither of  $\{a, b\}$  or  $\{c, d\}$  is equal to  $\{2\}$ , or one of these sets is equal to  $\{2\}$ . See Figure 3(ii). Suppose, without loss of generality, that  $a$  and  $b$  are both 2. Then we are able, by letting  $A$  be the annulus separating the central arc labeled  $n$  of the figure from the region  $R_2$ , to form an orbifold torus by attaching the boundary of a disk in  $\mathcal{S}$  that contains  $\{a, b\}$  to  $\partial A$ . Since  $\mathcal{Q}$  is atoroidal, this orbifold torus would bound a solid orbifold torus, forcing  $n = 1$  (so that that arc is not properly a part of the singular locus), the two supposed annuli to be parallel, and  $R_1$  to be equal to a solid orbifold torus. The other case, in which neither of  $\{a, b\}$  or  $\{c, d\}$  is equal to  $\{2\}$  and  $n \geq 1$ , may occur.

Note that, in the cases considered in the above paragraph, there can be no part of the singular locus that surrounds the part of the singular locus in Figure 3(ii), as this would give rise to an essential torus in  $\mathcal{Q}$  similar to the one illustrated in Figure 5.

Finally, there is the possibility that  $A$  is nonorientable and has either one singular point or two corner points (there are two such annuli, listed in Figure 1). In this instance, the boundary of a regular neighborhood of  $A$  falls into the category of Figure 3(ii), and just as in the case of a nonorientable annulus in Figure 3(i), the boundary of this regular neighborhood separates the regular neighborhood from the rest of  $\mathcal{Q}$  as an  $I$ -bundle. This completes the proof of the lemma.  $\square$

**3.1. Proof of the main theorem.** What follows is a proof of Theorem 1.1. Recall that  $\mathcal{O}$  is a compact, orientable, irreducible, turnover-reduced, 3-orbifold with underlying space  $S^3$  whose interior admits a hyperbolic structure of finite volume. Let  $\mathcal{S}$  be a closed, incompressible 2-suborbifold of  $\mathcal{O}$  of the form  $\mathcal{S} = S^2(n_1, n_2, n_3, n_4)$ . Our goal is a lower volume bound on  $\mathcal{O}$  in terms of the topology of  $\mathcal{S}$ . The plan of the proof is to find all possibilities for  $\text{guts}(\mathcal{S})$  by applying Lemma 3.1 and then applying Theorem 2.5.

*Proof.* To begin, we note that  $\partial\mathcal{O} = P \cup H$ , where  $P$  is the set of rigid cusp neighborhood boundaries and  $H$  the set of hyperbolic turnovers (corresponding to Euclidean and hyperbolic vertices in the singular set  $\Sigma$  of  $\mathcal{O}$ , respectively). The presence of hyperbolic turnovers in the boundary prevents the interior of  $\mathcal{O}$  from having a finite volume hyperbolic structure, but  $\mathcal{O}$  does admit a unique hyperbolic structure of finite volume in which the turnovers are realized as totally geodesic boundary components. Let  $D(\mathcal{O}) = \mathcal{O} \cup_H \overline{\mathcal{O}}$  be the double of  $\mathcal{O}$  along its totally geodesic turnover boundary components where  $\overline{\mathcal{O}}$  denotes  $\mathcal{O}$  with reversed orientation. The orbifold  $\mathcal{O}$  is naturally a 3-suborbifold of  $D(\mathcal{O})$ . Since an incompressible annulus in  $\mathcal{O}$  can be made to be

disjoint from any incompressible turnover in the boundary of  $\mathcal{O}$ ,  $D(\mathcal{O})$  has a hyperbolic structure of finite volume on its interior with possible cusps coming from Euclidean vertices in the singular set of  $\mathcal{O}$ . Let  $G = guts(D(\mathcal{O}) \setminus D(\mathcal{S}), D(p))$ . We then define  $guts(\mathcal{S})$  to be the intersection of  $G$  with  $\mathcal{O}$ . Note that we may apply Theorem 2.5 directly to  $guts(\mathcal{S})$  since  $\text{Vol}(G) = 2\text{Vol}(guts(\mathcal{S}))$ .

Let  $\mathcal{Q}$  denote one of the two components of  $\mathcal{O} \setminus \mathcal{S}$ ; it is a compact, orientable, irreducible, atoroidal, turnover-reduced 3-orbifold, with one incompressible boundary component  $\mathcal{S}$  (that is not a hyperbolic turnover) and all other boundary components consisting of rigid and hyperbolic turnovers. Note that  $\mathcal{Q}$  satisfies the assumptions of Lemma 3.1. We will abuse notation and call the collection of rigid boundary components  $P$ . Recall the notation  $\partial_0 \mathcal{Q} = \partial \mathcal{Q} - P$ . Because hyperbolic turnovers are always incompressible,  $\partial_0 \mathcal{Q}$  is incompressible. Using the remarks after Definition 2.2, we will identify the characteristic subpair and, subsequently, the guts of  $(\mathcal{Q}, P)$ .

If  $(\mathcal{Q}, \partial \mathcal{Q}_0)$  contains no essential annuli (that are not boundary parallel), then it is acylindrical and admits a hyperbolic metric with totally geodesic boundary. In particular, it is equal to its guts, and in this case, Theorem 2.5 provides a lower bound of

$$(3.1) \quad \text{Vol}(\mathcal{Q}) \geq -V_8 \chi(guts(\mathcal{S})) = -\frac{V_8}{2} \chi(\partial \mathcal{Q}) \geq -\frac{V_8}{2} \chi(\mathcal{S}).$$

The latter inequality takes into account that some components of the boundary may be hyperbolic turnovers, which we discard in the estimate because they may or may not be present. Otherwise,  $\mathcal{Q}$  admits one of the configurations of essential annuli from the Lemma 3.1. We examine the cases.

In the case of Figure 3(i), the essential annulus cuts off a solid torus with an annulus in its boundary contained in  $\partial \mathcal{Q}$ . The remaining piece could be pared acylindrical, in which case the Euler characteristic of its boundary, being the same as that of  $\mathcal{S}$ , yields the same lower volume bound as in (3.1). If not, then because it is not a solid orbifold torus, it must be an  $(I\text{-bundle}, \partial I\text{-subbundle})$  pair as in item 2.3.(3). In particular, we obtain no lower volume bound, but we do have a concrete description of the type of  $\mathcal{Q}$  in this case (and the next). See Proposition 4.1 below, after the proof. The case of Figure 3(i) when  $e = 2$  is exactly analogous to the previous case, with the same possible volume bounds.

In the case of Figure 3(ii), there are either one or two essential annuli, each of which is a disk with two singular points of order 2. If  $n \geq 2$ , then there are two annuli, and they divide  $\mathcal{Q}$  into three components. If  $n = 1$ , then there is only one annulus and it divides  $\mathcal{Q}$

into two components. In the first case, the “middle” component is a solid torus and so contributes nothing to the guts, and the “upper” and “lower” components may or may not contribute to the guts, depending on whether they are ( $I$ -bundle,  $\partial I$ -subbundle) pairs. If, for example, the upper component in the figure is not such a bundle, then it provides the following guts-based lower volume bound contribution, based on the fact that its boundary is a 2-orbifold  $X$  with underlying space the 2-sphere and with four singular points of orders  $a$ ,  $b$ , 2, and 2:

$$(3.2) \quad \text{Vol}(\mathcal{Q}) \geq -\frac{V_8}{2}\chi(X) = \frac{V_8}{2} \left(1 - \frac{1}{a} - \frac{1}{b}\right).$$

A similar computation holds for the lower component. In the case when both of these components are  $I$ -bundles, we do not obtain a volume bound, but we do have a characterization given in Proposition 4.1 below, after the proof.

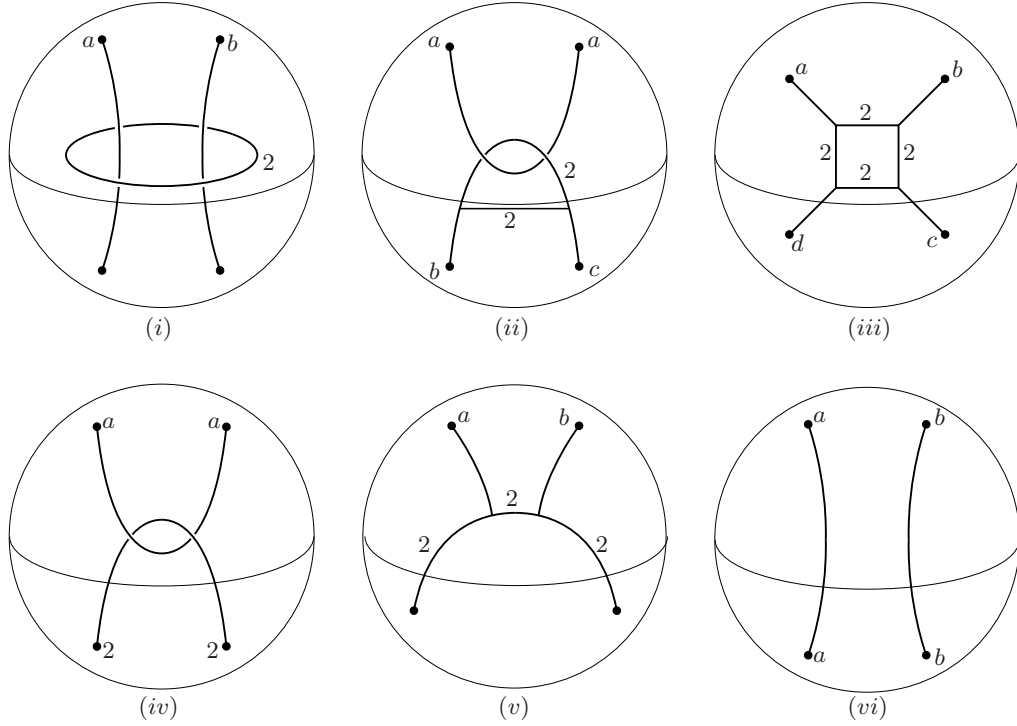
Combining these results, we see that the volume bounds we obtain from (3.1) and (3.2) fall into six categories, corresponding to the items in Theorem 1.1.1:

- (1) Both sides of  $\mathcal{S}$  in  $\mathcal{O}$  contribute fully to the guts, so we combine the two bounds of type (3.1), or
- (2) one side of  $\mathcal{S}$  contributes fully to the guts, and the other side contributes the guts only using two of the four singular points of  $\mathcal{S}$ , so we combine the bound of type (3.1) with the bound of type (3.2), or
- (3) one side of  $\mathcal{S}$  contributes fully to the guts, and the other side contributes nothing to the guts, so we use only one volume bound of type (3.1), or
- (4) each side of  $\mathcal{S}$  contributes to the guts using only two of the four singular points of  $\mathcal{S}$ , and so we combine two volume bounds of type (3.2), or
- (5) one side of  $\mathcal{S}$  contributes to the guts using only two of the four singular points of  $\mathcal{S}$ , and so we use only one volume bound of type (3.2), or
- (6) neither side of  $\mathcal{S}$  contributes to the guts, and we use Lemma 4.1 in Section 4.1 to classify these orbifolds.

This completes the proof of Theorem 1.1.  $\square$

#### 4. ORBIFOLD $I$ -BUNDLES AND HUNGRY ORBIFOLDS

In the proof of the main theorem, the only cases in which a lower bound could not be obtained involved the possibilities when the regions  $R$ ,  $R_1$ , and  $R_2$  in Figure 3 were ( $I$ -bundle,  $\partial I$ -subbundle) pairs. The following proposition classifies these orbifolds.

FIGURE 6. The different possibilities for  $I$ -bundles.

**Proposition 4.1.** *Let  $M$  be an orientable 3-orbifold with underlying space a 3-ball such that  $\partial M$  has underlying space a 2-sphere with 4 singular points, obtained from the orbifold  $\mathcal{Q}$  as above. If  $M$  is an orbifold  $I$ -bundle, then  $M$  has one of the forms given in Figure 6.*

*Proof.* For the purposes of this lemma, an orbifold  $I$ -bundle is a space of the form  $(F \times [-1, 1]) / ((x, y) \sim (\varphi(x), -y))$ , where  $F$  is a connected, orientable 2-orbifold and  $\varphi$  is an involution of  $F$  [6, page 443]. Of course, one possibility is that  $F$  has two singular points and  $\varphi$  is the identity, in which case  $M$  is just a product of  $F$  with an interval, as in Figure 6(vi). If the underlying space of  $F$  has genus greater than zero, then such a quotient as above will not have underlying space the 3-ball, because a component of its boundary will not have the appropriate underlying space. If the underlying space of  $F$  is the 2-sphere, then there are four choices (up to isotopy) for the involution  $\varphi$ : the identity, a reflection across a great circle, a rotation of order two, or the antipodal map.

Without even needing to consider the singular locus, we note that the identity map, the antipodal map, and a rotation of order two will each yield a quotient space either whose underlying space is not the 3-ball or that contains a 2-orbifold that cannot occur in  $\mathcal{Q}$  (by the construction



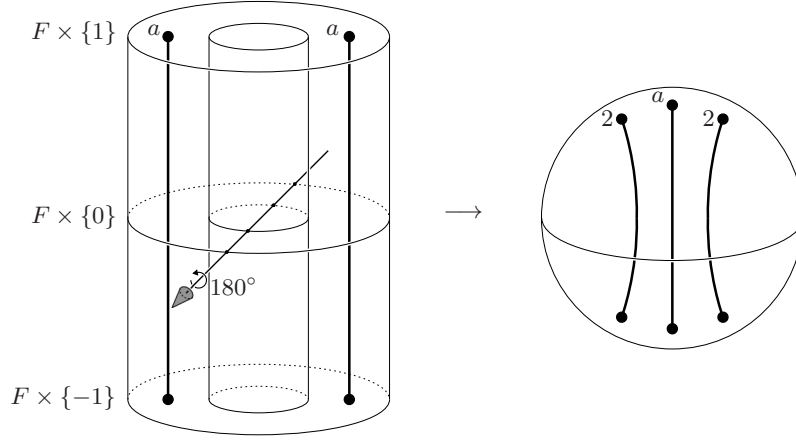


FIGURE 7. Symmetry of a surface crossed with an interval, yielding an  $I$ -bundle.

of  $\mathcal{Q}$ , as coming from the 3-sphere with an embedded trivalent graph). However, if  $F$  has underlying space the 2-sphere and if  $\varphi$  is a reflection across a great circle, then the quotient has underlying space a 3-ball and contains one unknotted loop labeled 2 in its interior. In order to obtain a quotient whose boundary contains four singular points, we have three options: (1) If  $F$  has four singular points of orders  $a$ ,  $a$ ,  $b$ , and  $b$ , and  $\varphi$  is a reflection that interchanges the singular points of corresponding orders, then the resulting quotient is depicted in Figure 6(i); (2) If  $F$  has four singular points of orders  $a$ ,  $a$ ,  $b$ , and  $c$ , and  $\varphi$  fixes only the singular points labeled  $b$  and  $c$  and interchanges the singular points labeled  $a$ , then the result is depicted in Figure 6(ii); (3) If  $F$  has four singular points of orders  $a$ ,  $b$ ,  $c$ , and  $d$  and  $\varphi$  fixes them all, then the result is Figure 6(iii).

If  $F$  has genus zero and multiple boundary components, then an involution that interchanges two boundary components will create higher genus in the boundary of the quotient, so  $\varphi$  must leave the boundary circles of  $F$  invariant. There are only three possibilities for how an involution of  $F$  can act on a boundary circle: the identity, a reflection, or the antipodal map. If  $F$  has more than one boundary circle, then the identity involution will create higher genus in the boundary of the quotient, and any action by the antipodal map will yield a quotient with a non-orientable crosscap in its boundary, contrary to our assumption on  $M$ . So we are left with a reflection, which extends from the boundary circles of  $F$  to a reflection of the whole orbifold  $F$ .

If  $\varphi$  is a reflection of a 2-orbifold  $F$  with genus zero, no singular points, and two boundary components, then the  $I$ -bundle quotient obtained from  $\varphi$  has four singular points of order two in its boundary. See Figure 7 (with  $a = 1$ ). If  $F$  has more than two boundary components, or at least two boundary components and some singular points, then the resulting quotient will have more than four singular points in its boundary. See Figure 7 (with  $a > 1$ ) in the case that  $F$  has two boundary components and one singular point. In the cases that we are considering, this is not possible. So  $F$  must have a single boundary component. Because  $\varphi$  is a reflection, we have only two possibilities: (1) If  $F$  has two singular points of the same order that are exchanged, then the result is Figure 6(iv); (2) If  $F$  has two singular points labeled  $a$  and  $b$  that are fixed by  $\varphi$ , then the result is Figure 6(iv). This completes the proof of the lemma.  $\square$

**4.1. The case of hungry orbifolds.** When the guts of  $\mathcal{O} \setminus \mathcal{S}$  are empty, then  $\mathcal{O}$  can have only one of finitely many forms of a certain type. We describe these types in this section here.

A *rational tangle operation* on a 2-sphere with four marked points is an isotopy of the 2-sphere that permutes the four marked points.

Let  $\mathcal{T}$  denote  $S^2$  with four marked points. Let  $\sigma : \mathcal{T} \times [0, 1] \rightarrow \mathcal{T}$  be an isotopy and let  $\sigma_t(x) = \sigma(x, t)$  for  $x \in \mathcal{T}$  and  $t \in [0, 1]$ . The *isotopy cylinder* for  $\sigma$  is the set

$$\mathcal{T}_\sigma = \{(\sigma_t(x), t) \mid x \in \mathcal{T}, t \in [0, 1]\}.$$

Note that in  $\mathcal{T}_\sigma$ , the marked points trace out a braid in  $\mathcal{T} \times I$ .

Suppose  $\mathcal{Q}_i$  is an orbifold with  $\mathcal{T}_i \subseteq \partial \mathcal{Q}_i$  for  $i \in \{0, 1\}$  where each  $\mathcal{T}_i$  is homeomorphic to  $\mathcal{T}$ . If  $\sigma$  is a rational tangle operation, then we can glue  $\mathcal{Q}_0$  to  $\mathcal{Q}_1$  along the isotopy cylinder for  $\sigma$  to obtain  $\mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1$ . More precisely,

$$\mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1 = (\mathcal{Q}_0 \amalg \mathcal{Q}_1 \amalg \mathcal{T}_\sigma) / \sim$$

where  $\sim$  is an identification such that  $x \sim (x, i)$  for each  $x$  in  $\mathcal{T}_i$  and  $i \in \{0, 1\}$ .

**Proposition 4.2.** *Let  $\mathcal{O}$  be a 3-orbifold containing an incompressible 2-suborbifold  $\mathcal{S}$  of the form  $S^2(n_1, n_2, n_3, n_4)$  or  $D_*^2(n_1, n_2)$ . Suppose that  $\text{guts}(\mathcal{S})$  is empty. Then  $\mathcal{O}$  has one of the following forms:*

- (1) *If  $\mathcal{S} = S^2(n_1, n_2, n_3, n_4)$ , then  $\mathcal{O} = \mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1$  where each of  $\mathcal{Q}_i$ ,  $i \in \{0, 1\}$  is one of the orbifolds in Figure 3 with each of  $R$ ,  $R_1$ , and  $R_2$  equal to one of the bundles from Figure 6, and  $\sigma$  a rational tangle isotopy.*

- (2) If  $\mathcal{S} = D_*^2(n_1, n_2)$ , then  $\mathcal{O} = \mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1$  where  $\mathcal{Q}_0$ , is one of the orbifolds in Figure 3 with each of  $R$ ,  $R_1$ , and  $R_2$  equal to one of the bundles from Figure 6 and  $\mathcal{Q}_1$  is a regular neighborhood of  $\mathcal{S}$  in  $\mathcal{O}$ , and  $\sigma$  a rational tangle isotopy.

*Proof.* Suppose that  $\mathcal{S} = S^2(n_1, n_2, n_3, n_4)$ . Since  $\mathcal{S}$  is orientable,  $\mathcal{O} \setminus \mathcal{S}$  has two components. Denote these components by  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$ . If  $\mathcal{Q}_i$  contains an incompressible annulus, then by Lemma 3.1,  $\mathcal{Q}_i$  and its annulus are configured as in Figure 3. By the characteristic suborbifold theory, each of the regions  $R$ ,  $R_1$ , and  $R_2$  in these cases must actually be an orbifold  $I$ -bundle (see Figure 6), for otherwise, the portion of  $\mathcal{Q}_i$  on that side of the essential annulus would contradict the fact that  $\text{guts}(\mathcal{S})$  is empty.

If  $\mathcal{Q}_i$  contains no essential annuli, then since  $\text{guts}(\mathcal{S})$  is empty, by the characteristic suborbifold theory,  $\mathcal{Q}_i$  must be an  $I$ -bundle. By Proposition 4.1,  $\mathcal{Q}_i$  must be of one of the six types described by Figure 6. By inspection, observe that  $I$ -bundles of types (i), (ii), and (iii) of Figure 6 contain an essential orbifold annulus. Therefore  $\mathcal{Q}_i$  must be an  $I$ -bundle of type (iv), (v), or (vi). However, each of these types have compressible boundaries, contradicting the assumption that  $\mathcal{S}$  is incompressible in  $\mathcal{O}$ .

Let  $\mathcal{Q}_i, i \in \{0, 1\}$  be two orbifolds as described in (1) of the statement of the proposition. If  $\sigma$  is any rational tangle operation on  $\mathcal{S}$ , then since  $\sigma$  is an isotopy, the two components of  $\mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1 \setminus \mathcal{S}$  have the same orbifold type as  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$ .

If  $\mathcal{S} = D_*^2(n_1, n_2)$ , then let  $\mathcal{Q}_1$  be a closed regular neighborhood of  $\mathcal{S}$  in  $\mathcal{O}$ . Then the boundary of  $\mathcal{Q}_1$  has the form  $S^2(n_1, n_1, n_2, n_2)$ . The same argument as in the orientable case can be used to show that  $\mathcal{O} = \mathcal{Q}_0 \sqcup_\sigma \mathcal{Q}_1$  where  $\mathcal{Q}_0$  is as described in the statement of the proposition and  $\sigma$  is a rational tangle operation on  $S^2(n_1, n_1, n_2, n_2)$ .

□

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